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Subscription of shares

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ABSTRACT

The paper studies share *subscription* schemes and claims that these schemes are useful when it is desirable to let an item be held or a project be undertaken by a group of bidders. Examples of such instances include the privatization of state-owned enterprises, the introduction of strategic investors, the procurement of large government construction projects, the issuance of treasury bills, and the resolution of the banks' toxic assets. I show that these share subscription schemes result in sale prices that *do* approach the fundamental value as the number of bidders increases. This is in contrast to share auctions in Wilson (1979) in which the Nash-equilibrium sale price *can* yield a significantly lower sale price than a unit-auction.

1. Introduction

The paper studies share *subscription* schemes and claims that these schemes are useful when it is desirable to let an item be held or a project be undertaken by a group of bidders. Examples of such instances include the privatization of state-owned enterprises, the introduction of strategic investors, the procurement of large government construction projects, the issuance of treasury bills, and the resolution of the banks' toxic assets.

It is useful to start with a simple example with two bidders that illustrates the difference between a share subscription scheme and a share auction. Following Wilson (1979), let us look at the case in which the two bidders know the value of the sale item, v, with certainty. In share auction, each bidder i submits a schedule of bids for shares, $x_i(p)$. And the sale price is the one that equates the demand and supply of shares. In share subscription, each bidder i subscribes a value v_i, and receives $v_i/(v_1+v_2)$ shares, with v_1+v_2 the total sale price. The difference between share auction and share subscription is that in the former case, there is no restriction on the functional form of $x_i(p)$; in the latter case, the functional form of $x_i(p)$ is restricted to be v_i/p and each bidder is free to choose v_i optimally.

In this paper, I discuss two types of subscription schemes: Nash Subscription scheme (NS) in which all bidders submit a sealed-subscription and Sequential Subscription scheme (SS) in which each bidder observes the values subscribed by the bidders before her. I analyze the equilibrium strategy in both NS and SS when the value of the item is known with certainty and I show that with more than two bidders, the sale price under SS strictly exceeds that under NS. I show that these share subscription schemes result in sale prices that *do* approach the fundamental value as the number of bidders increases. This is in contrast to share auctions in Wilson (1979) in which the Nash-equilibrium sale price *can* yield a significantly lower sale price than a unit-auction.

[1] When the value of the item is uncertain, the analysis of equilibrium strategy in subscription schemes is much more difficult, especially under SS. Nevertheless, a closed-form example may be possible along the line in Wilson (1979, pp 680–86) and is left for future research.

[2] Interestingly, the sale price, $v/2$, obtained by Wilson (1979, pp 677–78) is only one of infinitely many Nash equilibria: the sale price can be $\alpha v/(1+\alpha)$ for any $\alpha>0$. To see this in the case of n bidders, it suffices to show that

$$x(p) = \frac{1-(\frac{1+\alpha}{\alpha})^{\alpha}p^{\alpha}\,/\,(nv^{\alpha})}{n-1}$$

is an optimal strategy for all the bidders. This is straightforward and omitted ($\alpha=1$ coincides with Wilson's claim). Wilson (1979, pp 679) does recognize the possibility of a continuum of Nash equilibria in the case that allows uncertainty about the value and risk aversion among the bidders. Hence, the sale price could be arbitrarily close to zero in share auction. Of course, if the functional forms of bid schedule are *restricted* to be $x_i(p)=\delta_i-\gamma_i p^{\alpha}$ with high α, the seller can obtain $\alpha v/(1+\alpha)$ at Nash equilibrium, arbitrarily close to v. Nevertheless, the required functional forms are not as "natural" as v_i/p in the subscription schemes.

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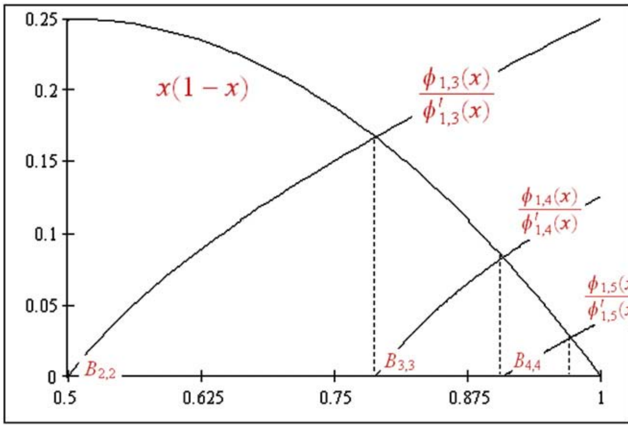

Fig. 1. The process of $B_{N,N}$ approaching unity.

One could draw an analogy between NS and Nash–Cournot Oligopoly game and between SS and Stackelberg Oligopoly game. However, we must understand that share subscription is neither quantity competition, nor price competition; it is value competition (price times quantity). We need to be aware of this when making comparison between the results obtained here and those in the Oligopoly literature.

Bernheim and Whinston (1986) propose first-price menu auctions to sell a divisible item. They show that the sale price is higher than in share auction. However, they do not require the auctioneer to set a price to clear the demand—the auctioneer can allocate the shares based on the menu of bids at hand to maximize the revenue. Since some bidders may get less than the amount they asked for if the bidders submit off-equilibrium menus, it induces more aggressive behavior at the equilibrium and the menu auction leads to higher revenue for the seller than share-auction. In other words, Bernheim and Whinston allow more freedom on the part of the seller and I put more restrictions on the part of the bidders, both leading to a better deal for the seller.

The rest of the paper is organized as follows. In Section 2, I examine Nash subscription (NS) scheme with known value in which the optimal strategy is easy to characterize. In Section 3, I characterize the optimal strategy in a sequential subscription (SS) scheme. The optimal strategy is highly non-linear. Nevertheless, a recursive but analytical representation is obtained. Section 4 concludes with discussions on unit auction, share auction, and share subscription.

2. Nash subscription scheme

There is an item with known value of *unity*, which needs to be sold to N bidders. In an NS, bidder i submits a sealed bid, $b_i \in [0,1]$, knowing that she will be allocated a fraction of the item equal to $b_i/\Sigma_{j=1}^N b_j$. The total value obtained by the seller is $\Sigma_{j=1}^N b_j$.

I assume that the bidders are risk neutral and bidder i would choose b_i to maximize

$$\frac{b_i}{\sum_{j=1}^N b_j} - b_i, \tag{1}$$

taking the values subscribed by other bidders $j \neq i$ as given (hence the name Nash Subscription). The objective function above is concave in b_i and thus the first order condition below is a necessary and sufficient condition for an interior solution:

$$\frac{\sum_{j=1}^N b_j - b_i}{\left[\sum_{j=1}^N b_j\right]^2} - 1 = 0.$$

In equilibrium, $b_j = b$ for any j.[3] Thus the first order condition above yields:

$$b = \frac{(N-1)}{N^2}.$$

The revenue for the seller is Nb, which equals $(N-1)/N$. The item is thus sold at a discount of $1/N$. The discount approaches to zero as N increases without bound.

The NS game could be interpreted as a special case of the Tullock rent-seeking game in which player i's expected payoff is

$$\pi_i(x_1, x_2, \ldots, x_n) = v\frac{x_i^R}{\sum_{j=1}^n x_j^R} - x_i$$

where the prize, v, is constant and $R>0$. If we denote this family of Tullock games as $T(v, R, n)$ as in Baye and Hoppe (2003), the NS game is mathematically identical to $T(1, 1, N)$ *ex ante*. *Ex post*, the Tullock game has only one winner, but all the bidders in the NS game get a piece. The other feature to note is that the optimal strategy of x_i for any i would be proportional to the prize v, hence it is without loss of generality to normalize v to unity.

Baye and Hoppe (2003) also establish the link between the Tullock game and a patent race game *à la* Loury (1979). Hence, our NS game here can also have the interpretation of a patent race game in which the interest rate approaches zero and the hazard rate takes the form: $h(x_i) = x_i$. As the interest rate approaches zero, a patent race game converges to its limiting static form. This limiting patent race game, the Tullock rent-seeking games, and our NS scheme may represent well the situations in which the players' moves are simultaneous or sequential but private. On the other hand, there are cases in which the players' moves may be sequential, sometimes partially detectable, other times fully revealed as required by government regulations. One could think, for example, about the rules involving transparency on presidential campaign contributions. It is therefore of interest to study sequential subscription games or sequential Tullock rent-seeking games.

3. Sequential Subscription

Sequential subscriptions are much harder to analyze. In an SS, the bidders are randomly ordered and the bids are openly submitted sequentially, hence the name: Sequential Subscription. It should be stressed that each bidder in the SS scheme submits a single bid when his turn comes up with no possibility of revising or withdrawing his bid at a later stage. The following analysis is done by backward induction. To fix notations, let b_i denote the value subscribed by bidder i, and B_i be the sum of values subscribed by the first i bidders. Any rational bidder, i, will make sure that $B_i \in (0,1)$.

Let N denote the number of bidders in the game. The Nth bidder's optimal strategy is easy to derive and can be characterized as $B_N = \sqrt{B_{N-1}}$, which is strictly increasing in B_{N-1}. Obviously, $B_N = B_N(B_{N-1}) = \ldots = B_N(B_{N-1}(\ldots(B_i))) \triangleq \Omega_i(B_i)$ for any $i = 1, 2, \ldots, N-1$. In general, it can be shown that $\Omega_i(B_i)$ is strictly increasing in $B_i \in (0,1)$, an intuitive result. Thus it is meaningful to write $B_i = \Omega_i^{-1}(B_N) \triangleq \phi_i(B_N)$ with $B_N \in (\lim_{B_i \to 0^+} \Omega_i(B_i), 1)$.

I now proceed to find the functional form of $\phi_i(\cdot)$. To begin with, we know:

$$B_{N-1} = \phi_{N-1}(B_N) = B_N^2.$$

More generally, we have the following Iteration Principle.

[3] The opportunity cost of funds and the liquidity constraints are abstracted away so the bidders are symmetric.

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3.1. The Iteration Principle

Let N be the numbers of bidders in the sequential subscription scheme, then $\phi_{N-1}(x)=x^2$ and if $N\geq 3$,

$$\phi_i(x) = \phi_{i+1}(x) - x(1-x)\phi'_{i+1}(x), \text{ for } i = 1, ..., N-2.$$

Proof. We already established that $\phi_{N-1}(x)=x^2$. The decision problem for bidder $i+1$ can be written as follows:

$$\max_{B_{i+1}} \frac{B_{i+1}-B_i}{B_N(B_{N-1}(...(B_{i+1})))} - [B_{i+1}-B_i].$$

The first order condition is:

$$\frac{B_N - [B_{i+1}-B_i]\frac{dB_N}{dB_{i+1}}}{B_N^2} - 1 = 0,$$

which yields

$$B_i = B_{i+1} - (B_N - B_N^2)\frac{1}{\Omega'_{i+1}(B_{i+1})}.$$

Note that $\Omega'_{i+1}(B_{i+1})\phi'_{i+1}(B_N)\equiv 1$ by definition, we arrive at:

$$\phi_i(x) = \phi_{i+1}(x) - x(1-x)\phi'_{i+1}(x).$$

The value of B_N will be pinned down in the fundamental equation below, which is the special case of the Iteration Principle recognizing that $B_0=0$.

3.2. The fundamental equation

B_N can be obtained from the following "fundamental equation":

$$B_N(1-B_N) = \phi_1(B_N) / \phi'_1(B_N).$$

From the Iteration Principle, we can easily derive $\phi_{N-2}(x)=x^2(2x-1)$. But when N is large, $\phi_1(x)$ quickly becomes a complicated polynomial of order N. Nevertheless we are able to establish the following propositions.

Proposition 1. The vanishing discount

The discount for selling the item vanishes as the number of bidder increases toward infinity: with two bidders, $1-B_2=1/2$ and with N bidders where $N\geq 3$, $1-B_N<\frac{1-\sqrt{1-2^{3-N}}}{2}$.

The proof is given in the Appendix. Comparing to the discount in the Nash Subscription scheme where the discount equals $1/N$, the discount in the Sequential Subscription scheme converges to zero much faster. When $N=10$, the discount is 10% under NS scheme but less than 0.2% under SS scheme. Fig. 1 depicts the process of B_N approaching unity.[4]

Proposition 2. The declining profit

For all bidders, the profit for bidder i, $\pi_i(B_N)=(1-B_N)^2\phi'_i(B_N)$, declines monotonically as N increases.

Proof. The profit for bidder i,

$$\pi_i(B_N) = \frac{B_N(1-B_N)\phi'_i(B_N)}{B_N} - B_N(1-B_N)\phi'_i(B_N)$$
$$= (1-B_N)^2\phi'_i(B_N)$$

Differentiating with respect to B_N gives:

$$\pi'_i(B_N) = (1-B_N)^2\phi''_i(B_N) - 2(1-B_N)\phi'_i(B_N)$$

Note the following inequality[5]:

$$B_N(1-B_N)\phi''_i(B_N) = 2B_N\phi'_i(B_N) - \phi'_{i-1}(B_N) < 2B_N\phi'_i(B_N)$$

Hence,

$$\pi'_i(B_N) = (1-B_N)^2\phi''_i(B_N) - 2(1-B_N)\phi'_i(B_N)$$
$$< (1-B_N)^2\frac{2B_N}{B_N(1-B_N)}\phi'_i(B_N) - 2(1-B_N)\phi'_i(B_N)$$
$$= 0$$

As N increases, we know that B_N increases and thus the profit of each bidder decreases.

The result in Proposition 2 is consistent with the notion that more competition lowers the profit for each competitor.

Proposition 3. The Law of Bisection

When N approaches infinity, the first bidder's bid and share approach 1/2, the second bidder's approach 1/4, the third bidder's approach 1/8, etc.[6]

The proof is given in the Appendix. This is a powerful result and could be empirically tested in rent-seeking or patent race games if the moves in these games are more sequential in nature.

Proposition 4. The first mover advantage

There is a first mover advantage when $N\geq 3$: the first bidder obtains the highest share and profit.

This proposition can be easily verified by numerical calculation when N is low and is proved more generally for $N>6$ in the Appendix.

Although the first mover gets the largest share of the profit, she may be willing to give up this advantage in exchange for an NS game. Take $N=3$ as an example. In an SS, applying the fundamental equation with $\phi_1(B_3)=B_3^2(2B_3-1)$, we obtain $B_3=0.789$ and the bids are: $b_1=0.359$, $b_2=0.263$ and $b_3=0.167$. The profit for the first bidder is $0.359/0.789-0.359=0.096$, which is lower than 1/9, her profit if an NS game is played.

When $N=2$, applying fundamental equation with $\phi_1(B_2)=B_2^2$, we find that $b_1=b_2=1/4$, thus there is no first mover advantage in this case. These bids are the same as in an NS scheme. This is a peculiar result because one would expect the bids to be different under NS and SS, given the analogy in the theory of duopoly that the quantities produced by the duopolies are different under the Nash–Cournot game and the Stackelberg game (see Boyer and Moreaux, 1986; and Anderson and Engers, 1992).

[4] When we discuss about what happens when the number of bidders, N, changes, it is important to use $b_{i,N}$, $B_{i,N}$ and $\phi_{i,N}$ to avoid confusion; the first subscript, i, indicates the bidder and the second subscript, N indicates the number of bidders in the game. Thus, $B_{N,N}=b_{1,N}+...+b_{N,N}$.

[5] Although $\phi_0(x)$ has no economic meaning here, its mathematical form is valid, namely it is a polynomial of order $N+1$ and by construction, $\phi_0(B_N)=0$ and $\phi'_0(B_N)>0$. Thus, the inequality also works for $i=1$.

[6] In Stackelberg oligopoly games, Boyer and Moreaux (1986) and Anderson and Engers (1992) show that a linear demand curve induces a "law of bisection." In a setting with incomplete information studied by Zhang and Zhang (1999), they proved a variant of this law. In their notation,

$$\frac{dq_{n+1}}{dq_n} = -\frac{1}{2}.$$

Another intriguing result is that the share of the first bidder is *not* a monotonically decreasing function of N. When $N=2$, we see that the first bidder's share is 1/2. The first bidder's share then decreases to $0.359/0.789=0.455$ when $N=3$. But from the Law of Bisection above, the first bidder's share rises later and approaches 1/2 again. Thus it is not monotonic.

What drives the non-monotonicity result? Note that the share of the first bidder is b_1/B_N; both the numerator and denominator are increasing functions of N. Intuitively, the profit is determined by the profit margin and the share. When N is small, it is more important for the first bidder to safeguard the profit margin and therefore he does not bid too aggressively. Thus as N increases but remains small, the first bidder would be willing to allow his share to slip somewhat to prevent significant erosion of the profit margin. On the other hand, when N goes above 4, the profit margin is already thin and the first bidder might as well grab a higher share and thus bid aggressively.

4. Discussions on Unit Auction, Share Auction, and Share Subscription

If it is desirable for the seller or for the society to allocate a divisible item to a number of bidders, as in the cases of privatization of state owned enterprises, introduction of foreign strategic investors, procurement of large government construction projects, issuance of treasury bills, or resolution of banks' toxic assets, the single unit auction is not an option. Wilson (1979) argues that share auction may result in a selling price that is too low to be considered. In fact, he could have made a stronger case that the selling price can be arbitrarily close to zero, as illustrated in the Introduction. This paper suggests that share subscriptions, which are equivalent to putting a natural restriction on the functional forms of the schedule of the bids, may allow the seller to allocate the shares to a number of bidders at a reasonable selling price. Indeed, the seller can control the discount of the selling price to the fundamental value by choosing the number of bidders.

There have been much concern in transitional economies during their reform stage when state assets are sold way below their "fundamental" value. In China, for example, the introduction of foreign strategic investors to China Construction Bank, the Bank of China, and Industrial and Commercial Bank of China were criticized by the public on the ground that the strategic investors were given excessively lucrative deals.

In principle, the issuance of government bonds or auction of foreign exchanges can be carried out using share subscription schemes. The current practices in the world are mainly two types: either each bidder pays its own bid price (sealed-bid discriminatory auction) or each bidder pays the same price (sealed-bid uniform-price auction, the same as the share auction studied in Wilson, 1979). There is a debate on which format will bring in higher expected revenue and empirical evidence seems to be mixed (see Umlauf, 1993 for a study of Mexican Treasury auctions and Tenorio, 1993 for a study of Zambian foreign exchange auctions). The argument against discriminatory auction is that it discourages relatively uninformed bidders because of the severity of the winner's curse, so bidding becomes concentrated among a few bidders who may be able to collude. The argument against uniform auction is given in Wilson (1979) and is articulated more clearly in Back and Zender (1993) as a different possibility for collusion by bidders who submit high inframarginal bids (steep demand curve) that inhibit competition. The share subscription schemes presented here retain the uniform-price feature (all bidders pay the same price per share) and hence get rid of the winner's curse, and at the same time deprive the bidders from submitting a steep demand curve (in fact forcing the demand elasticity to be unity).

Even an IPO can be conducted, with the help of the Internet, using sequential subscription schemes. Sequential subscription would certainly have brought in more revenue for the listing company, given the numerous instances when the prices of the listing company rise to be several folds of the IPO price (Baidu.com and Focus Media Holdings 2005). Most popular IPO practice nowadays shares some features of subscription schemes. First, the listing company typically gives a price range. Sometimes the price range could be rather wide. Investors submit the number of shares they would like to subscribe without knowing the exact IPO fixing price. The fixing price is determined ex-post once all the subscriptions are received. When the IPO is heavily oversubscribed, the price will typically be fixed at the upper limit of the price range and the shares will be rationed to the subscribers. I would propose that this procedure be replaced by an online sequential subscription scheme, with the small subscribers first (no repeated subscription allowed and no withdraws allowed) and institutional investors the last. Essentially, the idea is that the small investors who subscribe first would have to trust that the institutional investors would not push the price too high. Since everyone pays the same price during this IPO stage, the institutional investors have no incentive to oversubscribe. Of course, once I make a distinction between small and large investors, I already step outside of the formal analysis presented above. One needs to formalize the concept of small investors by liquidity constraint, as perhaps in Che and Gale (1998).

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Appendix A

In the following, I first treat the number of bidders, N, as fixed.

Lemma A1. Let a series of functions defined on [0,1] be given as follows:

$$\tilde{\phi}_{N-2}(x) = 2x - 1$$

$$\tilde{\phi}_i(x) = (2x-1)\tilde{\phi}_{i+1}(x) - x(1-x)\tilde{\phi}'_{i+1}(x) \text{ for } i = 1, \ldots N-3.$$

Show that $\tilde{\phi}_i(x)=0$ has $N-i-1$ distinct roots in (0,1) for any $i=1,\ldots N-2$.

Proof. The proof here is in the same spirit as in Sturm's Theorem.

It is straightforward to verify that $\tilde{\phi}_i(x)$ is a polynomial of order $N-i-1$ with a positive coefficient on x^{N-i-1}.

Let us now show the result by backward induction. First of all, the result trivially holds for $i=N-2$. Suppose the result holds for all $i \geq k+1$. We need to show that it also holds for $i=k$.

Given that the result holds for $i=k+1$, it means that $\tilde{\phi}_{k+1}(x)$ can be written as $\tilde{\phi}_{k+1}(x)=A_{k+1}(x-r_1)\cdots(x-r_{N-k-2})$ where $0<r_1<r_2<\ldots<r_{N-k-2}<1$ and A_{k+1} is a positive constant.

$$\begin{aligned}\tilde{\phi}_k(x) &= (2x-1)\tilde{\phi}_{k+1}(x) - x(1-x)\tilde{\phi}'_{k+1}(x) \\ &= (2x-1)A_{k+1}(x-r_1)\cdots(x-r_{N-k-2}) \\ &\quad -x(1-x)A_{k+1}\sum_{j=1}^{N-k-2}\prod_{\substack{1\leq m\leq N-k-2 \\ m\neq j}}(x-r_m)\end{aligned}$$

Clearly,

$$\tilde{\phi}_k(1) = A_{k+1}(1-r_1)\cdots(1-r_{N-k-2}) > 0$$

$$\tilde{\phi}_k(r_{N-k-2}) = -r_{N-k-2}(1-r_{N-k-2})(r_{N-k-2}-r_1)\cdots(r_{N-k-2}-r_{N-k-3}) < 0.$$

More generally,

$$\begin{aligned}sign(\tilde{\phi}_k(r_{j+1})) &= sign(-(r_{j+1}-r_1)\cdots(r_{j+1}-r_j)(r_{j+1}-r_{j+2})\cdots(r_{j+1}-r_{N-k-2})) \\ &= sign(-(-1)^{N-k-2-(j+2)+1}) \\ &= sign((-1)^{N-k-j}),\end{aligned}$$

and thus would switch sign between $\tilde{\phi}_k(r_{j+1})$ and $\tilde{\phi}_k(r_j)$. Furthermore, $\tilde{\phi}_k(r_1)$ has a sign equal to sign$((-1)^{N-k})$ and $\tilde{\phi}_k(0)=-A_{k+1}(-r_1)\cdots(-r_{N-k-2})$ has a sign equal to sign$((-1)^{N-k-1})$. This shows that $\tilde{\phi}_k(x)$, which is a polynomial of order $N-k-1$, changes sign consecutively at points 0, $r_1,\ldots r_{N-k-2}$,1. This implies that $\tilde{\phi}_k(x)$ has $N-k-1$ distinct roots in (0,1).

Lemma A2. Let a series of functions defined on [0,1] be given as follows:

$$\phi_{N-2}(x) = x^2(2x-1)$$

$$\phi_i(x) = \phi_{i+1}(x) - x(1-x)\phi'_{i+1}(x) \quad \text{for} \quad i = 1,\ldots N-2.$$

Show that $\phi_i(x)=x^2\tilde{\phi}_i(x)$, where $\tilde{\phi}_i(x)$ is defined in Lemma A1.

Proof. Trivial and omitted.

Lemma A3. $\phi_1(x)/\phi'_1(x)$ is an increasing function of x in $(r_{\max},1]$ where $r_{\max}$ is the maximum of the roots of $\phi_1(x)$ in (0,1).

Proof. From Lemma A1 and A2 we have, with a slight abuse of notation:

$$\phi_1(x) = x^2A_1(x-r_1)\cdots(x-r_{N-2}),$$

where $r_{N-2}=r_{\max}$.

Thus,

$$\frac{\phi_1(x)}{\phi'_1(x)} = \frac{x^2A_1(x-r_1)\cdots(x-r_{N-2})}{2xA_1(x-r_1)\cdots(x-r_{N-2}) + x^2A_1\sum_{j=1}^{N-2}\prod_{\substack{1\le m\le N-2\\ m\neq j}}(x-r_m)}$$

$$= \frac{1}{2x^{-1} + \sum_{j=1}^{N-2}(x-r_j)^{-1}} \text{ for any } x\in(r_{\max},1]$$

which increases in interval $(r_{\max},1]$.

In general, if $f(x)$ is a polynomial and if all roots of $f(x)=0$ are real [7], then $f(x)/f'(x)$ is increasing in any interval that contains no root of $f(x)=0$. Whether the roots are distinct or repetitive is not important. In Lemma A1, we show that all roots of $\tilde{\phi}_i(x)=0$ are real by showing that they are all in (0,1). Lemma A1 then implies that all the roots of $\phi_i(x)=0$ are also real.

Strictly speaking, the polynomial $\phi_i(x)$ and $\tilde{\phi}_i(x)$ should be denoted as $\phi_{i,N}(x)$ and $\tilde{\phi}_{i,N}(x)$ because they implicitly depend on the number of bidders, N. More specifically, $\tilde{\phi}_{N-2,N}(x)=2x-1$ and $\tilde{\phi}_{N-1,N+1}(x)=2x-1$. In fact, we have the following result.

Lemma A4. $\phi_{i,N}(x)\equiv\phi_{i+1,N+1}(x)$ and $\tilde{\phi}_{i,N}(x)\equiv\tilde{\phi}_{i+1,N+1}(x)$ for $i=1,\ldots N-2$.

Proof. Trivial and omitted.

Lemma A5. Let the largest root of $\phi_{1,N+1}(x)=0$ be denoted by $r_{\max,N+1}$. Then, $r_{\max,N+1}$ is the solution to $x(1-x)=\phi_{1,N}(x)/\phi'_{1,N}(x)$.

Proof. $\phi_{1,N+1}(r_{\max,N+1})=0$ implies $\tilde{\phi}_{1,N+1}(r_{\max,N+1})=0$, which implies:

$$(2r_{\max,N+1}-1)\tilde{\phi}_{2,N+1}(r_{\max,N+1}) \\ -r_{\max,N+1}(1-r_{\max,N+1})\tilde{\phi}'_{2,N+1}(r_{\max,N+1}) = 0.$$

Making use of Lemma A4 yields,

$$(2r_{\max,N+1}-1)\tilde{\phi}_{1,N}(r_{\max,N+1}) \\ -r_{\max,N+1}(1-r_{\max,N+1})\tilde{\phi}'_{1,N}(r_{\max,N+1}) = 0.$$

[7] If a polynomial $f(x)$ has imaginary roots, for example, $f(x)=x^2+1$, $f(x)/f'(x)$ may not always be increasing.

Rewrite the above equality in terms of $\phi_{1,N}(\cdot)$ by noting that $\phi_{1,N}(x)=x^2\tilde{\phi}_{1,N}(x)$, we have:

$$r_{\max,N+1}(1-r_{\max,N+1}) = \frac{\phi_{1,N}(r_{\max,N+1})}{\phi'_{1,N}(r_{\max,N+1})}.$$

Lemma A6. B_N is given by $r_{\max,N+1}$.

Proof. Strictly speaking, B_N should be denoted as $B_{N,N}$ with the second subscript indicating the number of bidders in the game and the first subscript indicating the Nth bidder. Namely, $B_{i,N}$ denotes the sum of the first i bids in an N-bidder subscription scheme. We will establish that $B_{N,N}=r_{\max,N+1}$, $N=2,3,\ldots$

Again, we use induction. For $N=2$, $r_{\max,3}$ is the solution to $\phi_{1,3}(x)=0$.

$$\phi_{1,3}(x) = x^2(2x-1)$$

Thus, $r_{\max,3}=1/2$, which coincides with our result in Section 3: $B_{2,2}=1/2$.

Suppose it is true that $B_{N,N}=r_{\max,N+1}$. Clearly, from the constructive proof of Lemma A1, we know that $r_{\max,N+2}$ is the unique solution to $\tilde{\phi}_{1,N+2}(x)=0$ between $(r_{\max,N+1},1)$. Lemma A5 says that $r_{\max,N+2}$ solves the fundamental equation:

$$x(1-x) = \frac{\phi_{1,N+1}(x)}{\phi'_{1,N+1}(x)}$$

Hence $r_{\max,N+2}$ is the $B_{N+1,N+1}$ that we are looking for. This completes the proof by induction.

Lemma A7. $\phi_{1,N}(1)=\ldots=\phi_{N-2,N}(1)=1$ and $\phi'_{i,N}(1)=2^{N-i}$ and $\phi''_{i,N}(1)=18\times3^{N-i-2}-2^{N-i+1}$ for $i=1,\ldots N-2$.

Proof. From the definition,

$$\phi_{N-2,N}(x) = x^2(2x-1),$$

$$\phi_{i,N}(x) = \phi_{i+1,N}(x) - x(1-x)\phi'_{i+1,N}(x),\ i = 1,\ldots N-3,$$

and the fact that $\phi_{1,N}(x)$ is a polynomial for all i, it is easy to see that:

$$\phi_{1,N}(1) = \ldots = \phi_{N-2,N}(1) = 1.$$

Also, we have:

$$\phi'_{N-2,N}(1) = 2^2,\ \phi''_{N-2,N}(1) = 10$$

and

$$\phi'_{i,N}(x) = 2x\phi'_{i+1,N}(x) - x(1-x)\phi''_{i+1,N}(x).$$

Thus,

$$\phi'_{i,N}(1) = 2\phi'_{i+1,N}(1) = \ldots = 2^{N-i-2}\phi'_{N-2,N}(1) = 2^{N-i}.$$

Furthermore,

$$\phi''_{i,N}(x) = 2\phi'_{i+1,N}(x) + (4x-1)\phi''_{i+1,N}(x) - x(1-x)\phi'_{i+1,N}(x).$$

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This gives rise to:

$$\begin{aligned}\phi''_{i,N}(1) &= 2\phi'_{i+1,N}(1) + 3\phi''_{i+1,N}(1)\\ &= 2^{N-i} + 3\phi''_{i+1,N}(1)\\ &= 2^{N-i} + 3(2^{N-i-1} + 3\phi''_{i+2,N}(1))\\ &= \ldots\\ &= \sum_{j=0}^{N-i-3} 3^j 2^{N-i-j} + 3^{N-i-2}\phi''_{N-2,N}(1)\\ &= 2^{N-i}\frac{(3/2)^{N-i-2}-1}{1/2} + 3^{N-i-2}\times 10\\ &= 18\times 3^{N-i-2} - 2^{N-i+1}.\end{aligned}$$

Proof of Proposition 1. The vanishing discount

Note that,

$$\begin{aligned}B_{N,N}(1-B_{N,N}) &= \frac{\phi_{1,N}(B_{N,N})}{\phi'_{1,N}(B_{N,N})}\\ &< \frac{\phi_{1,N}(1)}{\phi'_{1,N}(1)}, \text{ by Lemma A3}\\ &= \frac{1}{2^{N-1}}.\end{aligned}$$

Combining this inequality and the knowledge that $B_{N,N}>1/2$ for $N\geq 3$, we arrive at:

$$B_{N,N} > \frac{1+\sqrt{1-2^{3-N}}}{2},$$

thus we proved that for $N\geq 3$,

$$1-B_N < \frac{1-\sqrt{1-2^{3-N}}}{2}. \quad \square$$

From Proposition 1, we could see that $\lim_{N\to\infty} B_{N,N}=1$. Furthermore, Proposition 1 indicates the convergence speed. For example, we can easily see that when $N=10$, $B_{N,N}>0.99804$.

Lemma A8. $\phi'_{i,N-1}(x)/\phi'_{i,N}(x)$ is a decreasing function of x in interval $[B_{N-1,N-1},1]$.

Proof.

$$\begin{aligned}\frac{\phi'_{i,N-1}(x)}{\phi'_{i,N}(x)} &= \frac{\phi'_{i,N-1}(x)}{2x\phi'_{i+1,N}(x) - x(1-x)\phi''_{i+1,N}(x)}\\ &= \frac{\phi'_{i,N-1}(x)}{2x\phi'_{i,N-1}(x) - x(1-x)\phi''_{i,N-1}(x)}\\ &= \frac{1}{2x - x(1-x)\frac{\phi''_{i,N-1}(x)}{\phi'_{i,N-1}(x)}}.\end{aligned}$$

Applying the same line of proofs as in Lemma A1 and A2 on $\phi'_{i,N-1}(x)$, we can easily show that $\phi'_{i,N-1}(x)$ has the following properties in interval $[B_{N-1,N-1},1]$: (1) $\phi'_{i,N-1}(x)\geq 0$, $\phi''_{i,N-1}(x)\geq 0$, and $\phi'''_{i,N-1}(x)\geq 0$; and (2) $\phi'_{i,N-1}(x)/\phi''_{i,N-1}(x)$ increasing in x. Note that $x(1-x)$ is decreasing in x in interval $[B_{N-1,N-1},1]$ as $B_{N-1,N-1}\geq 1/2$. Thus $\phi'_{i,N-1}(x)/\phi'_{i,N}(x)$ is decreasing in x in interval $[B_{N-1,N-1},1]$.

Proof of Proposition 3. The Law of Bisection

From the definition:

$$\phi_{i,N}(x) = \phi_{i+1,N}(x) - x(1-x)\phi'_{i+1,N}(x),\ i=1,\ldots N-3$$

we see that:

$$\phi'_{i,N}(x) = 2x\phi'_{i+1,N}(x) - x(1-x)\phi''_{i+1,N}(x).$$

Hence:

$$\phi'_{i+1,N}(B_{N,N}) = \frac{\phi'_{i,N}(B_{N,N})}{2B_{N,N}} + \frac{(1-B_{N,N})\phi''_{i+1,N}(B_{N,N})}{2}$$

Using the equation above to substitute out $\phi'_{i+1,N}(B_{N,N})$ below:

$$\begin{aligned}&\phi_{i+1,N}(B_{N,N}) - \phi_{i,N}(B_{N,N})\\ &= B_{N,N}(1-B_{N,N})\phi'_{i+1,N}(B_{N,N})\\ &= B_{N,N}(1-B_{N,N})\left[\frac{\phi'_{i,N}(B_{N,N})}{2B_{N,N}} + \frac{(1-B_{N,N})\phi''_{i+1,N}(B_{N,N})}{2}\right]\\ &= \frac{\phi_{i,N}(B_{N,N}) - \phi_{i-1,N}(B_{N,N})}{2B_{N,N}} + \frac{B_{N,N}(1-B_{N,N})^2}{2}\phi''_{i+1,N}(B_{N,N}).\end{aligned}$$

Note that $0<\phi''_{i+1,N}(B_{N,N})<\phi''_{i+1,N}(1)$ (implied from property (1) in the proof of Lemma A8). We have

$$\begin{aligned}&\frac{\phi_{i,N}(B_{N,N}) - \phi_{i-1,N}(B_{N,N})}{2B_{N,N}}\\ &\leq \phi_{i+1,N}(B_{N,N}) - \phi_{i,N}(B_{N,N}) \qquad (2)\\ &\leq \frac{\phi_{i,N}(B_{N,N}) - \phi_{i-1,N}(B_{N,N})}{2B_{N,N}} + \frac{B_{N,N}(1-B_{N,N})^2}{2}\phi''_{i+1,N}(1)\end{aligned}$$

But for the second term on the right side of the inequality, we have

$$\begin{aligned}&0 < (1-B_{N,N})^2\phi''_{i+1,N}(1)\\ &< \left(\frac{1-\sqrt{1-2^{3-N}}}{2}\right)^2 (18\times 3^{N-i-3} - 2^{N-i}) \qquad (3)\\ &< 2^{4-2N}(18\times 3^{N-i-3} - 2^{N-i}).\end{aligned}$$

The limit of the right-hand side is equal to zero as N goes to infinity, which implies that $(1-B_{N,N})^2\phi''_{i+1,N}(1)$ converges to zero. Thus,

$$\lim_{N\to\infty}[\phi_{i+1,N}(B_{N,N}) - \phi_{i,N}(B_{N,N})] = \frac{1}{2}\lim_{N\to\infty}[\phi_{i,N}(B_{N,N}) - \phi_{i-1,N}(B_{N,N})]$$

which means that as $N\to\infty$, it must be true that,

$$b_2 = \frac{1}{2}b_1 \text{ and } b_{i+1} = \frac{1}{2}b_i.$$

But given that $B_{N,N}\to 1$ as implied by Proposition 1, we must have:

$$b_1 + \frac{1}{2}b_1 + \frac{1}{2^2}b_1 + \ldots + \frac{1}{2^i}b_1 + \ldots = 1.$$

Thus,

$$b_1 = 1/2 \text{ and } b_i = \frac{1}{2^i}. \quad \square$$

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The difficulty in proving the Law of Bisection is that $\phi_{i,N}(\cdot)$ depends on N. This requires careful treatment when taking the limit. For example, we see that

$$\lim_{N\to\infty} \phi_{1,N}(B_{N,N}) = \frac{1}{2},$$

despite the fact that $\phi_{1,N}(1) = 1$ for any given N and $\lim_{N\to\infty} B_{N,N} = 1$, an intriguing result.[8]

Proof of Proposition 4. The first mover advantage

It is straightforward to verify that $b_{i+1,N}/b_{i,N} < 1$ is true for N between 3 and 7. Suppose that $b_{i+1,N}/b_{i,N} < 1$ is true for $N = K \geq 7$, then we see that for $N = K+1$ and for any $i \geq 2$,

$$\begin{aligned}
\frac{b_{i+1,K+1}}{b_{i,K+1}} &= \frac{\phi_{i+1,K+1}(B_{K+1,K+1}) - \phi_{i,K+1}(B_{K+1,K+1})}{\phi_{i,K+1}(B_{K+1,K+1}) - \phi_{i,K}(B_{K+1,K+1})} \\
&= \frac{\phi'_{i,K}(B_{K+1,K+1})}{\phi'_{i,K+1}(B_{K+1,K+1})} \\
&< \frac{\phi'_{i,K}(B_{K,K})}{\phi'_{i,K+1}(B_{K,K})} \text{ (Lemma A8)} \\
&= \frac{\phi'_{i-1,K-1}(B_{K,K})}{\phi'_{i-1,K}(B_{K,K})} \\
&= \frac{b_{i,K}}{b_{i-1,K}} \\
&< 1, \text{ by backward induction assumption.}
\end{aligned}$$

What remains to be shown is that $b_{2,K+1}/b_{1,K+1} < 1$.

$$\begin{aligned}
\frac{b_{2,K+1}}{b_{1,K+1}} &< \frac{\phi'_{1,K}(B_{K,K})}{\phi'_{1,K+1}(B_{K,K})} \\
&= \frac{1}{2B_{K,K} - B_{K,K}(1-B_{K,K})\frac{\phi''_{1,K}(B_{K,K})}{\phi'_{1,K}(B_{K,K})}}.
\end{aligned} \tag{4}$$

We need to show that the denominator of the right-hand side of Eq. (4) is greater than 1.

Note the following inequality, which comes from Eqs. (2) and (3):

$$b_{i+1,N} < \frac{b_{i,N}}{2B_{N,N}} + \frac{1}{2}\left(\frac{1-\sqrt{1-2^{3-N}}}{2}\right)^2 (18 \times 3^{N-i-3} - 2^{N-i}).$$

Summing over $i = 1,\ldots N-1$, we have:

$$\begin{aligned}
B_{N,N} - b_{1,N} &< \frac{B_{N-1,N}}{2B_{N,N}} + \frac{1}{2}\left(\frac{1-\sqrt{1-2^{3-N}}}{2}\right)^2 \left(18 \times 3^{-2}\frac{3^{N-1}-1}{2} - 2(2^{N-1}-1)\right) \\
&= \frac{B_{N,N}}{2} + \frac{1}{2}\left(\frac{1-\sqrt{1-2^{3-N}}}{2}\right)^2 ((3^{N-1}-1) - (2^N-2)) \\
&\leq \frac{B_{N,N}}{2} + \frac{1}{2}\left(\frac{1-\sqrt{1-2^{3-7}}}{2}\right)^2 ((3^{7-1}-1) - (2^7-2)) \text{ for any } N \geq 7.
\end{aligned}$$

where the last inequality is due to the fact that $\left(\frac{1-\sqrt{1-2^{3-N}}}{2}\right)^2 \times ((3^{N-1}-1) - (2^N-2))$ is a decreasing function in N. Thus for $N \geq 7$, we have:

$$\begin{aligned}
b_{1,N} &> \frac{B_{N,N}}{2} - 0.075877 \\
&\geq \frac{B_{7,7}}{2} - 0.075877 \\
&\geq 0.41618.
\end{aligned}$$

Therefore, for $K \geq 7$, the denominator of Eq. (4), as shown below, is indeed greater than unity:

$$\begin{aligned}
& 2B_{K,K} - B_{K,K}(1-B_{K,K})\frac{\phi''_{1,K}(B_{K,K})}{\phi'_{1,K}(B_{K,K})} \\
&= 2B_{K,K} - B^2_{K,K}(1-B_{K,K})^2\frac{\phi''_{1,K}(B_{K,K})}{\phi_{1,K}(B_{K,K})} \\
&\geq 2B_{7,7} - \frac{(1-B_{K,K})^2\phi''_{1,K}(1)}{b_{1,K}} \\
&\geq 2B_{7,7} - \left(\frac{1-\sqrt{1-2^{3-K}}}{2}\right)^2 \frac{18 \times 3^{K-3} - 2^K}{0.41618} \\
&\geq 2 \times 0.984 - \left(\frac{1-\sqrt{1-2^{3-7}}}{2}\right)^2 \frac{(2 \times 3^6 - 2^7)}{0.41618} \\
&= 1.1624,
\end{aligned}$$

Thus, for any $K \geq 7$,

$$\frac{b_{2,K+1}}{b_{1,K+1}} < \frac{1}{1.1624} < 1.$$

[8] This is reminiscent of the following classic example. Let $f_n(x_n) = x_n^n$ where $x_n = 1 - 1/n$. Clearly $x_n \to 1$ as $n \to \infty$ and $f_n(1) = 1$ for any n. But

$$\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n = \frac{1}{e}.$$

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